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THE VIBRATION MODES AND EIGENFREQUENCIES OF CONICAL (AND CYLINDRICAL) SHELLS* /133**

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ABSTRACT. An exact solution of vibration modes and transverse eigenfrequencies of conical shells is studied. Donnell type differential equations with variable coefficients are used, neglecting the tangential components of inertia force.

In order to reduce the computing effort, a simplified method is recommended. When vibrating, the elastic restoring effects in the shell due to membrane extensional forces and bending (and twisting) moments are regarded as two springs in parallel. Then this element of shell is equivalent to the parallel springs system with single degree of freedom, and the whole shell can be represented by the sum of infinite systems of this single type, where the stiffness of springs are functions of the coordinates.

The analytical method presented in this paper can be used for shells with arbitrary conical angle α and various boundary conditions (the conditions at the vertex point of complete conical shell are discussed.)

In this paper the exact solution of the vibrational modes and transverse vibration eigenfrequencies of conical shells is found analytically. A simplified calculation method is suggested for practical purposes.

We use thin shell theory type differential equations or Donnell type differential equations of motion. Neglecting the tangential components of inertia forces, we deduce an uncoupled equation for the transverse displacement function, from which a power series solution for the vibrational mode is obtained. It is found that the vibrational mode of a conical shell possesses nonperiodic oscillatory characteristics and a rapidly increasing amplitude.

Owing to the complexity of the above calculations, a simplified method is suggested. The physical concept is as follows. Figure 2 shows the physical model constructed for an arbitrary element of the shell. The membrane tension caused by vibration and the restoring forces caused by bending moments are considered to be two springs with the spring constants k_1 and k_2. The element of the shell has the mass m. In this way we obtain a parallel spring system with a single degree of freedom. Therefore, the entire shell is equivalent to an infinite sum of these systems, where the stiffness of the spring is a function

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** Numbers in the margin indicate pagination in the original foreign text.

of the coordinates. The eigenfrequencies of this model can be calculated from a superposition of $\omega^2 = \frac{k_1}{m} + \frac{k_2}{m} = \omega_1^2 + \omega_2^2$, where ω_1 and ω_2 can be obtained independently from membrane theory and pure bending theory. The calculations are greatly simplified and satisfactory results were obtained. A comparison with experiments is made at the end of this paper.

The present theory can be applied on conical shells with any cone angle. As a check, we calculated the $\alpha = 0$ case, which is the circular cylinder. A simplified method to calculate the vibration of cylindrical shells is also presented. The treatment of the apex boundary conditions is also discussed.

SYMBOLS

a -- mean radius of cylindrical shell
E -- coefficient of elasticity (shell and material)
h -- shell thickness
g -- acceleration of gravity
ℓ -- height of cylindrical shell
n — eigenvalue of circumferential vibration of conical (cylindrical) shell (integer)
u, v, w -- longitudinal, circumferential, and lateral displacements
$U(x), V(x), W(x)$ -- vibrational mode function
x, y, z -- longitudinal, circumferential, and lateral coordinates of conical (cylindrical) shell
x_0 -- the distance measured along a generator between the vertex and the top (smaller end) of conical shell
x_1 -- the distance measured along a generator between the vertex and the bottom (larger end) of conical shell
α -- semi-vertex angle
θ -- circumferential angular coordinate of conical shell (positive direction counterclockwise)
μ -- Poisson ratio (shell material)
ρ -- density of shell material
ω -- eigenfrequency (circular frequency)

$$d = \nu^4 - \left(\frac{3\mu^2 + 10\mu - 1}{1 - \mu}\right)\frac{\nu^2}{2} + 1,$$

$$\lambda^2 = \sqrt{4e - 1} = \sqrt{2\nu^2\left[\frac{1 - 2\mu - 3\mu^2}{(1 - \mu)(1 - \mu^2)}\right] - 1},$$

$$e = \frac{\nu^2}{2}\left[\frac{1 - 2\mu - 3\mu^2}{(1 - \mu)(1 - \mu^2)}\right], \qquad \nu = \frac{n}{\sin\alpha},$$

$$p = \frac{h^2 \operatorname{tg}^2\alpha}{12(1 - \mu^2)}, \qquad \Omega^2 = \frac{\rho(1 - \mu^2)}{gE}\omega^2,$$

$$q = \frac{12\Omega^2}{h^2}, \qquad k^4 = \frac{12\rho(1 - \mu^2)}{h^2 gE}\omega^2;$$

Symbols for differential operators:

$$\nabla^2\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{1}{x}\frac{\partial}{\partial x} + \frac{1}{x^2\sin^2\alpha}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{1}{x}\frac{\partial}{\partial x} + \frac{1}{x^2\sin^2\alpha}\frac{\partial^2}{\partial\theta^2}\right),$$

$$\nabla_0^2\nabla_0^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{1}{a^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{1}{a^2}\frac{\partial^2}{\partial\theta^2}\right),$$

$$\nabla_x^2\nabla_x^2 = \left(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - \frac{\nu^2}{x^2}\right)\left(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - \frac{\nu^2}{x^2}\right),$$

$$\nabla_{0x}^2\nabla_{0x}^2 = \left(\frac{d^2}{dx^2} - \frac{n^2}{a^2}\right)\left(\frac{d^2}{dx^2} - \frac{n^2}{a^2}\right),$$

$$\Delta_0 = \frac{1}{x}\frac{d}{dx}, \qquad \Delta_0^2 = \frac{1}{x}\frac{d}{dx}\left(\frac{1}{x}\frac{d}{dx}\right), \cdots$$

$$\Delta = x\frac{d}{dx}, \qquad \Delta^2 = x\frac{d}{dx}\left(x\frac{d}{dx}\right), \cdots$$

$$L = [\Delta^4 - 2(\nu^2 + 1)\Delta^2 + d].$$

1. Introduction

In recent years authors of many countries investigated the problem of statics of conical shells. However, the dynamical problem is rarely studied, even for simple problems of line vibrations. As far as the author knows, Strutt (Ref. 1), in 1933, first studied the vibrations of a conical shell under the conditions of an undistorted middle plane. However, this is not consistent with the actual situation. In 1938 Federhofer (Ref. 2), based on Pflüger's (Ref. 3) stability analysis of conical shells, derived the general equations for the vibrations of a conical shell. He used a power function $x^2(1 - x/x_1)^2$ as an approximation to the vibration mode function, and sought solutions using Rayleigh's method. He also discussed the condition of the undistorted middle plane. After 1958 Trapezin (Ref. 4, 5) used trigonometric functions as an approximation to the vibration mode function (same as in the case of cylindrical shell), and used Galerkin's method to solve for axisymmetric vibrations of /135
the conical shell and other problems. In the same year, Herrmann and Mirsky (Ref. 6) made similar investigations. They also used a trigonometric function as the approximate vibration mode and used the Rayleigh-Ritz method to calculate eigenfrequencies. In addition, they discussed the axisymmetric vibrations under the assumptions of membrane theory, and introduced some solutions containing certain special conditions. Obviously, the solutions using trigonometric functions as the approximate vibration mode can only be applied to small cone angles (cone vertex angle $2\alpha < 30°$) and truncated conical shells. The reason for this is that the lateral vibration mode of a conical shell is not periodic as is the case for a circular cylindrical shell. In 1960 Alumyae (Ref. 7) investigated the axisymmetric vibration of a conical shell and discussed the characteristics and the integration of the differential equations, but he did not go into general cases. In conclusion, the research on problems of vibrating conical shells is still considered inadequate. There are also discrepancies in the calculations of existing work. We cannot present a more detailed discussion here because until now there exist neither experimental nor more accurate theoretical analyses.

We think that although very accurate results can be obtained by solving the equations directly, it can be accomplished only under an appropriate choice of the vibrational mode function. At present there exists no dependable experiment or theory upon which this choice of vibrational mode function can be based. Due to the mathematical similarities, we refer to the more thoroughly investigated works in the field of instability of conical shells. We should mention the works of Mushtari and Sachenkov (Ref. 8) who used an interchange of coordinates to obtain a more accurate approximate solution; Hoff (Ref. 9, 10) under the assumptions of small cone angles, used a simplified strain-displacement relationship (simpler than flat shell theory) to derive an independent equation for an exact solution, but the convergence of his series solution is rather poor, which limits its practical usefullness. Seide (Ref. 11, 12) went one step further than (Ref. 8) and worked out numerous solutions. He compared the forms and results of various approximate solutions of the stability problem. Other authors used $x^2 \sin(\pi x/\ell)$ power function as the form of approximate solution in stability calculations. This is also closer to the actual situation than using trigonometric functions.

The object of the present paper is to obtain a more accurate solution to the problem of the vibrational modes and eigenfrequencies of a conical shell. As we all know, it is very difficult to use analytical methods to solve the general vibrational equations (Ref. 2). The present paper utilizes the flat shell theory equations (Ref. 15, 16) (circular conical shell is the zero Gaussian curvature shell, this type of equation can be used when $n > 2$). We also neglect the tangential components of the inertial forces in the governing equations to facilitate computation[(1)]. From the viewpoint of order of magnitude analysis, those terms are indeed higher order terms compared with the terms due to lateral inertial forces and internal forces, and thus can be neglected. Besides, from the practical computational view point, we usually neglect these terms because the longitudinal and circumferential vibrational eigenfrequencies are much larger than the lateral eigenfrequency. Therefore, the above simplification is a reasonable one. After some routine calculations, the final general analytical solution for the vibration of the conical shell can be applied for various cone angles, irrespective of whether the cone is truncated or whole (but $n > 2$ has to hold). We also suggested the treatment of the boundary conditions at the vertex in the latter case. The results obtained can, of course, be also applied to the case of a conical shell plate.

Because the general analytical solution is too complicated, this method is useful in the study of vibrational modes, but not useful in engineering applications. Therefore, for practical purposes, we also introduced a simplified calculation method.

2. The Basic Equations for the Vibration of a Conical Shell

We adopt a coordinate system x, θ, z which denote, respectively, the generator direction, the circumferential direction, and the normal direction normal to the middle plane of the conical shell (a right-handed system). The corresponding displacements are denoted by u, v, w. Note that from now on, w is positive when directed inwards.

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(1) An independent equation can be derived without neglecting the inertial terms in the beginning, but neglecting some related small terms in the course of the differential operation. However, the calculations become very involved.

According to Hooke's law and the results from the integration of stress along the thickness of the shell (where we have neglected $z/(x \operatorname{tg} \alpha)$ in the integration and we shall discuss the vertex point later), we can obtain the relationships which hold between the internal forces, internal moments and strain, change of curvature and torsional rate from a consideration of the equilibrium of an elemental volume on the shell. Then, using the strain-displacement relationship derived by Love (Ref. 17), we can obtain a system of equations describing the general motion in terms of the displacements. However, we still cannot solve these differential equations with variable coefficients. Therefore, some simplification is necessary. Due to the accuracy required in practice, one can also make some simplifications, such as neglecting the effect of the tangential components of the displacements u, v on the change of curvature k_x, k_θ and on the torsional rate τ. In other words, using the conventional /136 assumptions of torque theory in practical engineering (flat shell theory [Ref. 15]) we can derive a simplified set of equations describing the general motion of the shell (frequently called Donnell-type equations):

$$\left.\begin{aligned}
&\frac{\partial^2 u}{\partial x^2}+\frac{1}{x}\frac{\partial u}{\partial x}-\frac{u}{x^2}+\left(\frac{1+\mu}{2}\right)\frac{1}{x\sin\alpha}\frac{\partial^2 v}{\partial x\,\partial\theta}-\left(\frac{3-\mu}{2}\right)\frac{1}{x^2\sin\alpha}\frac{\partial v}{\partial\theta}+\\
&\quad+\left(\frac{1-\mu}{2}\right)\frac{1}{x^2\sin^2\alpha}\frac{\partial^2 u}{\partial\theta^2}+\frac{w}{x^2\operatorname{tg}\alpha}-\frac{\mu}{x\operatorname{tg}\alpha}\frac{\partial w}{\partial x}=\frac{\rho(1-\mu^2)}{gE}\frac{\partial^2 u}{\partial t^2},\\
&\left(\frac{1-\mu}{2}\right)\frac{\partial^2 v}{\partial x^2}+\left(\frac{1-\mu}{2}\right)\frac{1}{x}\frac{\partial v}{\partial x}-\left(\frac{1-\mu}{2}\right)\frac{v}{x^2}+\left(\frac{1+\mu}{2}\right)\frac{1}{x\sin\alpha}\frac{\partial^2 u}{\partial x\,\partial\theta}+\\
&\quad+\left(\frac{3-\mu}{2}\right)\frac{1}{x^2\sin\alpha}\frac{\partial u}{\partial\theta}+\frac{1}{x^2\sin^2\alpha}\frac{\partial^2 v}{\partial\theta^2}-\frac{\cos\alpha}{x^2\sin^2\alpha}\frac{\partial w}{\partial\theta}=\frac{\rho(1-\mu^2)}{gE}\frac{\partial^2 v}{\partial t^2},\\
&\frac{\mu}{x\operatorname{tg}\alpha}\frac{\partial u}{\partial x}+\frac{u}{x^2\operatorname{tg}\alpha}+\frac{\cos\alpha}{x^2\sin^2\alpha}\frac{\partial v}{\partial\theta}-\frac{w}{x^2\operatorname{tg}^2\alpha}-\frac{h^2}{12}\nabla^2\nabla^2(w)=\frac{\rho(1-\mu^2)}{gE}\frac{\partial^2 w}{\partial t^2}.
\end{aligned}\right\}\qquad(1)$$

We shall discuss briefly the properties of this set of equations. Let us consider the two limits: (a) As $\alpha \to 0$, $x \to \infty$, $x \sin \alpha$ and $x \operatorname{tg} \alpha \to a$, the above equations reduce to the equations of vibration (Donnell type) for a circular cylindrical shell with radius a; (b) If we take $\alpha \to \pi/2$ and for x, let $x \sin \alpha \to r$, $x \operatorname{tg} \alpha \to \infty$, and if we utilize the conventional assumptions used in the calculations of the vibrations of circular plate, i.e., neglecting u, v, then the equations of vibration for a circular plate can be deduced from equation (1). Thus, we predict equation (1) can be applied to cases for any angle α. Note that using flat shell theory or Donnell type equations, the requirement is $\frac{h}{x \operatorname{tg} \alpha} \ll 1$, for example $\frac{h}{x \operatorname{tg} \alpha} < \frac{1}{30} - \frac{1}{20}$ (this is also the usual definition of a "thin" shell). This requirement is satisfied for truncated conical shells, but obviously cannot be satisfied at the vertex points of whole conical shells. Equation (1) cannot be valid, even in the vicinity of the vertex point. Further on, the problem of vibration discussed in this paper deals with properties of the whole shell, which is different from problems related to local stresses. From the energy view point, the kinetic and potential (strain) energy of the region near the vertex due to vibration is very small compared to the whole shell. Because the region near the vertex which does not satisfy the thin shell requirement is very small (its volume is about one percent of the whole) and due to the following calculations and experiments, the strain of the region near the vertex is also very small. We can say that the energy per volume is very small. Thus, in the calculations of vibrations, the above contradiction should not lead to very large errors in the solution, and we will apply equation (1) to the calculation of vibration of the whole shell. The above discussion does not apply for very small cone angles (such as $\alpha < 5° - 10°$) or very short whole conical shells. Usually these kinds of conical shells should be regarded as variable cross section columns, and column-like vibrations should be considered (the case of $n = 1$). These cases are out of the scope of the present paper. The following refers only to shell vibrating situations when $n > 2$.

3. Solution of the Vibration Equation

If we want to analytically solve the set of equations (1), an independent equation should first be derived. Th obtain this, we assume $w \gg v > u$ as is usually done in thin shell studies, and make the assumption that the tangential inertial forces can be neglected while the lateral inertial forces are retained. From the form of equations (1), it can be shown that the solution to u, v, w must be in the form of a series in $x^m \cos n\theta \sin \omega t$. Substituting this into the equation, we see that the relative magnitudes of the terms representing the internal forces can basically be divided into three levels of importance: m^2 or ν^2, m or ν, and 1. These levels were obtained from a comparison of the terms representing the same displacement (any one of u, v, w). Furthermore, if we consider the case $w \gg v, u$, the whole set of equations can be divided into many more levels of importance (4 - 5 levels). The importance of the inertial forces is represented by $\frac{\rho(1-\mu^2)}{gE}\omega^2 x_1^2$. This is always smaller than 1 if the lateral /137 eigenfrequency is not too high. At the same time, both ν and m have larger values, at least ν or $m > 5 - 6$ (usually they are both greater than 10). Therefore, the terms representing the tangential components of the inertial forces in equations (1) are indeed the highest order small terms, and can be neglected without noticeable errors in the solutions for the vibrational modes and the lateral eigenfrequencies.

In addition, in the calculations we substitute $(1 + \mu)$ for the coefficient $(3 - \mu)/2$ which appear in the first and second equations of the set of equations in (1). (We also point out that these terms were neglected by the small cone angle assumption in (Ref. 9). From the order of magnitude analysis, this does not seem to be reasonable.) When $\mu = 1/3$, this substitution is entirely correct. From solid state physics research, it follows that the Poisson ratio of common metals lies in the vicinity of 1/3 (aluminum, titanium or copper alloys). Note that the terms with the coefficient $(3-\mu)/2$ are of medium importance (ν level) in the equations. Thus, if there were slight discrepancies in the coefficients, the effect on the entire calculation is small (about the same error as in neglecting the tangential component of the inertia forces). Thus, even if the material property $\mu < 0.30$ or > 0.40, the above substitution is still allowable.

With the above assumptions, we also assume solutions of the form

$$\left.\begin{aligned} u &= U(x)\cos n\theta \sin \omega t, \\ v &= V(x)\sin n\theta \sin \omega t, \\ w &= W(x)\cos n\theta \sin \omega t, \end{aligned}\right\} \tag{2}$$

Then equations (1) reduce into the following set of equations:

$$\begin{aligned} &\Delta_0^2(x^2U) - \frac{2}{x^2}\Delta_0(x^2U) + \frac{3}{x^2}U + \left(\frac{1+\mu}{2}\right)\nu\cdot\Delta\left(\frac{V}{x^2}\right) - \\ &\quad - \left(\frac{1-\mu}{2}\right)\frac{\nu^2}{x^2}U = \frac{\mu}{x\,\mathrm{tg}\,\alpha}\frac{dW}{dx} - \frac{W}{x^2\,\mathrm{tg}\,\alpha}, \end{aligned} \tag{3a}$$

$$\begin{aligned} &\left(\frac{1-\mu}{2}\right)\left[\Delta^2\left(\frac{V}{x^2}\right) + 4\Delta\left(\frac{V}{x^2}\right) + 3\left(\frac{V}{x^2}\right)\right] - \\ &\quad - \left(\frac{1+\mu}{2}\right)\nu\cdot\frac{1}{x^2}\Delta_0(x^2U) - \frac{\nu^2}{x^2}V = -\frac{\nu}{x^2\,\mathrm{tg}\,\alpha}W, \end{aligned} \tag{3b}$$

$$\mu\Delta(U) + U + \nu V - \frac{W}{\mathrm{tg}\,\alpha} - \frac{h^2}{12}x^2\,\mathrm{tg}\,\alpha\cdot\nabla_x^2\nabla_x^2(W) = -\Omega^2x^2\,\mathrm{tg}\,\alpha\cdot W. \tag{3c}$$

Using the differential operator Δ on equation (3b) and substituting the value of $\Delta\left(\frac{V}{x^2}\right)$ from equation (3a), after some algebra we obtain the coupled equations for U and W (which is then multiplied by x^2). Then we multiply equation (3a) by x^4, differentiate with respect to Δ_0, and substitute the value of $\Delta_0(x^2U)$ into the equation. There result the following coupled equations for V and W:

$$\begin{aligned} L(U) &= \frac{\mu}{\mathrm{tg}\,\alpha}[\Delta^3(W) - \Delta(W)] - \frac{1}{\mathrm{tg}\,\alpha}[\Delta^2(W) - W] + \\ &\quad + \frac{\nu^2}{\mathrm{tg}\,\alpha}\Delta(W) - \left(\frac{2\mu}{1-\mu}\right)\frac{\nu^2}{\mathrm{tg}\,\alpha}W, \end{aligned} \tag{4a}$$

$$\begin{aligned} L(V) &= \left(\frac{1+\mu}{1-\mu}\right)\frac{\nu}{\mathrm{tg}\,\alpha}[\mu\Delta^2(W) - (1-2\mu)\Delta(W) - 2W] - \\ &\quad - \left(\frac{2}{1-\mu}\right)\frac{\nu}{\mathrm{tg}\,\alpha}[\Delta^2(W) - W] + \frac{\nu^3}{\mathrm{tg}\,\alpha}W. \end{aligned} \tag{4b}$$

After applying the operator L to equation (3c), we substitute equations (4a) and (4b), and note that

$$L[\Delta] = \Delta[L],$$

Then an independent (uncoupled) equation for the lateral vibrational mode function W(x) is obtained: /138

$$[\Delta^4 - \Delta^2 + e]W + \frac{h^2\,\mathrm{tg}^2\,\alpha}{12(1-\mu^2)}L[x^2\cdot\nabla_x^2\nabla_x^2(W)] = \frac{\Omega^2\,\mathrm{tg}^2\,\alpha}{(1-\mu^2)}L(x^2W). \tag{4c}$$

For this differential equation with variable coefficients we assume a power series type solution:

$$W(x) = x^r \sum_{s=0}^{\infty} C_s x^s = \sum_{s=0}^{\infty} C_s x^{r+s}. \tag{5}$$

Substituting the above equation into equation (4c), an indicial equation is obtained:

$$(r^2 - \nu^2)[(r-2)^2 - \nu^2][(r-2)^4 - 2(\nu^2+1)(r-2)^2 + d] = 0. \tag{6}$$

The roots are:

$$\begin{aligned} &r_1, r_2 = \pm \nu, \quad r_3, r_4 = \pm \nu + 2, \\ &r_5, r_6, r_7, r_8 = \pm \left[(\nu^2 + 1) \pm \nu \sqrt{\frac{3(1+\mu)^2}{2(1-\mu)}}\right]^{\frac{1}{2}} + 2. \end{aligned} \tag{7}$$

The values of r_j ($j = 1, 2, \ldots, 8$) are all real. We see that the magnitude of $|r_j|$ is comparable to the magnitude of ν (in general the value of ν is large), which agrees with the previous assumption that m and ν are of the same order of magnitude. Using the condition that the coefficients of each exponent s are zero, we obtain the coefficients C_{js} (the following are multiples of C_{j0}):

$$C_{j1} = C_{j3} = C_{j5} = C_{j7} = \cdots = C_{j,2s+1} = 0, \tag{8}$$

$$\begin{aligned} C_{js} = &-\frac{[(r_j+s-2)^4 - (r_j+s-2)^2 + e]}{p[(r_j+s)^2-\nu^2][(r_j+s-2)^2-\nu^2][(r_j+s-2)^4-2(\nu^2+1)(r_j+s-2)^2+d]} \cdot \\ &\cdot C_{j,s-2} + \frac{q}{[(r_j+s)^2-\nu^2][(r_j+s-2)^2-\nu^2]} C_{j,s-4} \\ &(s = 2, 4, 6, 8, 10, \cdots), \quad (j = 1, 2, \cdots, 8). \end{aligned} \tag{9}$$

Since $C_{j,-1} = C_{j,-2} = \ldots = 0$, we have

$$C_{j2} = -\frac{[r_j^4 - r_j^2 + e]}{p[(r_j+2)^2 - \nu^2][r_j^2 - \nu^2][r_j^4 - 2(\nu^2+1)r_j^2 + d]} C_{j0}, \tag{10}$$

where, in this equation, C_{j0} is an undetermined constant. Then the solution of W(x) can be written as

$$W(x) = \sum_{j=1}^{8} \sum_{s=0,2,4}^{\infty} C_{js} x^{r_j+s} = \sum_{j=1}^{8} C_j x^{r_j} f_j(x) = \sum_{j=1}^{8} C_j Z_j(x). \tag{11}$$

Later on we shall use C_j ($j = 1, 2, 3, \ldots, 8$) as an arbitrary constant. Therefore, the undetermined constant in $f_j(x)$ can be taken as 1 or any arbitrary value. Because the difference between the indicial roots r_1 and r_3, as well as between r_2 and r_4, is 2, the vibrational mode functions $Z_1(x)$ and $Z_3(x)$, $Z_2(x)$ and $Z_4(x)$ become linearly dependent. We must find two more independent solutions.

According to the general theories of power series solutions and considering the case $r_j = \pm\nu$, we assume

$$C_{j0} = d'_{j0}[r_j + (-1)^j \nu] \quad (j = 1, 2), \tag{12}$$

Using the symbol:

$$d_{js} = [r_j + (-1)^j \nu] \cdot \frac{C_{js}}{C_{j0}} \quad (j = 1, 2;\ s = 0, 2, 4, \cdots), \tag{13}$$

four independent solutions can be written down

$$\left.\begin{aligned} Z_1(x) &= Z_3(x) \log x + x^{\nu} d'_{10}[1 + d'_{12}x^2 + d'_{14}x^4 + \cdots + d'_{1s}x^s + \cdots], \\ Z_2(x) &= Z_4(x) \log x + x^{-\nu} d'_{20}[1 + d'_{22}x^2 + d'_{24}x^4 + \cdots + d'_{2s}x^s + \cdots], \end{aligned}\right\} \tag{14}$$

where

$$\begin{aligned} d'_{j2} = \frac{\partial}{\partial r_j}(d_{j2}) &= \Big[-2r_j(2r_j^2 - 1) + (r_j^4 - r_j^2 + e)\Big\{\frac{4r_j[r_j^2 - (\nu^2 + 1)]}{[r_j^4 - 2(\nu^2 + 1)r_j^2 + d]} + \\ &+ \frac{1}{[r_j - (-1)^j \nu]} + \frac{2(r_j + 2)}{[(r_j + 2)^2 - \nu^2]}\Big\}\Big] \div \\ &\div \{p[(r_j + 2)^2 - \nu^2][r_j - (-1)^j \nu][r_j^4 - 2(\nu^2 + 1)r_j^2 + d]\}, \end{aligned} \tag{15}$$

$$\begin{aligned} d'_{js} = \frac{\partial}{\partial r_j}(d_{js}) &= \Big\{\Big[-d'_{j,s-2} + d_{j,s-2}\Big\{\frac{4(r_j+s-2)[(r_j+s-2)^2-(\nu^2+1)]}{[(r_j+s-2)^4-2(\nu^2+1)(r_j+s-2)^2+d]} + \\ &+ \frac{2(r_j + s - 2)}{[(r_j + s - 2)^2 - \nu^2]} + \frac{2(r_j + s)}{[(r_j + s)^2 - \nu^2]}\Big\}\Big] \times \\ &\times [(r_j + s - 2)^4 - (r_j + s - 2)^2 + e] - \\ &- 2d_{j,s-2}(r_j + s - 2)[2(r_j + s - 2)^2 - 1]\Big\} \div \\ &\div \{p[(r_j + s)^2 - \nu^2][(r_j + s - 2)^2 - \nu^2][(r_j + s - 2)^4 - \\ &- 2(\nu^2 + 1)(r_j + s - 2)^2 + d]\} + \\ &+ q\Big[d'_{j,s-4} - d_{j,s-4}\Big\{\frac{2(r_j + s)}{[(r_j + s)^2 - \nu^2]} + \frac{2(r_j + s - 2)}{[(r_j + s - 2)^2 - \nu^2]}\Big\}\Big] \div \\ &\div \{[(r_j + s)^2 - \nu^2][(r_j + s - 2)^2 - \nu^2]\} \\ &(j = 1, 2), \quad (s = 4, 6, 8, 10, 12, \cdots); \end{aligned} \tag{16}$$

$$\left.\begin{aligned} Z_3(x) &= d'_{10}x^{\nu}[d_{12}x^2 + d_{14}x^4 + \cdots + d_{1s}x^s + \cdots], \\ Z_4(x) &= d'_{20}x^{-\nu}[d_{22}x^2 + d_{24}x^4 + \cdots + d_{2s}x^s + \cdots]. \end{aligned}\right\} \tag{17}$$

We can also write $Z_3(x) = x^{r_3}f_3(x)$ and $Z_4(x) = x^{r_4}f_4(x)$. Because equation (11) already contains the arbitrary constants C_j ($j = 1, 2, 3, \ldots, 8$), we shall make $d'_{10} = d'_{20} = 1$ and $C_{30} = C_{40} = C_{50} = \ldots = C_{80} = 1$ without any loss of generality.

Under certain conditions, ν may be an integer. $Z_3(x)$ and $Z_4(x)$ again become linearly dependent. In this case we must take

$$\begin{aligned} Z_3(x) &= d'_{40}x^{-\nu+2}[d_{4,2\nu-2}x^{2\nu-2} + d_{4,2\nu}x^{2\nu} + \cdots] \log x + \\ &+ d'_{40}x^{-\nu+2}[1 + d'_{42}x^2 + d'_{44}x^4 + \cdots + d'_{4s}x^s + \cdots] \end{aligned} \tag{18}$$

(when ν is an integer)

in which d'_{40} is determined by $C_{40} = d'_{40}(r_j + \nu - 2)$, where the procedure is the same as when $j = 1, 2$, but taking $j = 4$, $r_j = -\nu + 2$. When $s = 2\nu - 2$, 2ν, $2\nu + 2$, ..., substitute $(r_j + \nu - 2)$ for $[r_j + (-1)^j\nu]$ in equations (13) and (16). This is similar to the previous case of $s = 2, 4, 6, \ldots$ However, when $s = 2, 4, 6, \ldots, 2\nu - 4$, we should take $d'_{4s} = C_{4s}/C_{40}$. The rest is the same, and d'_{40} can also be assumed as unity.

When the Poisson ratio $\mu = \frac{1}{3}$, we see that $r_5, r_6, r_7, r_8 = \pm \nu + 3$, $\pm \nu + 1$, i.e., the indicial roots again differ by an integer 2. Because in the series solution we only take even values of s, the functions $Z_5(x), \ldots, Z_8(x)$ are still independent.

According to Cauchy's criteria we know at once the series for the $f_j(x)$ are convergent. However, the convergence is rather slow. Suppose we want to calculate the value of $f_j(x)$ to 4 or 5 significant digits. We must take 10, 16, and 24 terms respectively at $x = 2, 5, 10$. The characteristic exponents and variables of the series are very similer to Bessel functions for large arguments.

For example, in Figure 1, the curve for $f_3(x)$ shows the properties of /140
the vibrational mode function. For clarity this figure uses two sets of scales. The characteristic of the curve when $x > 5$ is similar to that for $x < 5$, which is a curve with variable period and increasing amplitude. (For instance, when $x = 5, 8, 10, \ldots, f_3(x) = -2446.2, -85872, + 767000, \ldots$). Because the amplitude increases so fast that it is inconvenient for plotting, we have omitted that part of the figure. From this we see that $f_j(x)$ is a non-periodic oscillatory function with rapidly increasing amplitude. Its amplitude would increase even faster after being multiplied by x^{r_j} (and when r_j is positive). These properties can also be referred to when choosing the pre-assumed vibration mode function in calculating the approximate solutions. However, the slow convergence of the series $f_j(x)$ greatly limits the practical usefulness of this kind of solution. There are two ways to solve this problem. One is to find an asymptotic expression from the mathematical viewpoint. The other is to simplify the calculations from purely physical concepts. This shall be introduced as follows.

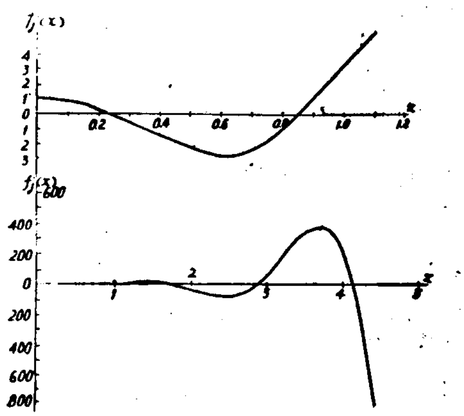

Figure 1.

From the lateral vibrational mode function W(x) and equations (4a) and (4b) we find

$$
\left.\begin{aligned}
U(x) &= \sum_{j=1}^{2} \sum_{s=0,2,4}^{\infty} (A_{js}\log x + A'_{js}) x^{r_j+s} + \sum_{j=3}^{8} \sum_{s=0,2,4}^{\infty} A_{js} x^{r_j+s} = \sum_{j=1}^{8} C_j X_j(x), \\
V(x) &= \sum_{j=1}^{2} \sum_{s=0,2,4}^{\infty} (B_{js}\log x + B'_{js}) x^{r_j+s} + \sum_{j=3}^{8} \sum_{s=0,2,4}^{\infty} B_{js} x^{r_j+s} = \sum_{j=1}^{8} C_j Y_j(x).
\end{aligned}\right\} \tag{19}
$$

When ν is an integer, the functions $X_3(x)$ and $Y_3(x)$ for $j = 3$ should contain the term log x as in the case when $j = 1, 2$. The coefficient of the above equation can be calculated from the following formula:

$$
\left.\begin{aligned}
A_{js} &= C_j \frac{\left[\mu(r_j+s)^3 - (r_j+s)^2 + (\nu^2-\mu)(r_j+s) - \left(\frac{2\mu}{1-\mu}\right)\nu^2 + 1\right]}{\mathrm{tg}\,\alpha[(r_j+s)^4 - 2(\nu^2+1)(r_j+s)^2 + d]} d_{js}, \\
B_{js} &= C_j \frac{\nu\left[-(2+\mu)(r_j+s)^2 - \frac{(1+\mu)(1-2\mu)}{(1-\mu)}(r_j+s) + \nu^2 - \left(\frac{2\mu}{1-\mu}\right)\right]}{\mathrm{tg}\,\alpha[(r_j+s)^4 - 2(\nu^2+1)(r_j+s)^2 + d]} d_{js} \\
&(j = 1, 2, 3, \cdots, 8), \quad (s = 2, 4, 6, 8, 10, \cdots)
\end{aligned}\right\} \tag{20}
$$

(Note: when ν is not an integer, for $j = 3, 4, \ldots, 8$, $d_{js} = C_{js}$)

$$
\left.\begin{aligned}
A'_{js} = C_j &\Bigg\{\Bigg[\mu(r_j+s)^3 - (r_j+s)^2 + (\nu^2-\mu)(r_j+s) - \\
&- \left(\frac{2\mu}{1-\mu}\right)\nu^2 + 1\Bigg] d'_{js} + [3\mu(r_j+s)^2 - 2(r_j+s) + \\
&+ (\nu^2-\mu)]d_{js} - 4(r_j+s)[(r_j+s)^2 - (\nu^2+1)] \times \\
&\times \frac{\left[\mu(r_j+s)^3 - (r_j+s)^2 + (\nu^2-\mu)(r_j+s) - \left(\frac{2\mu}{1-\mu}\right)\nu^2+1\right]}{[(r_j+s)^4 - 2(\nu^2+1)(r_j+s)^2 + d]} d_{js}\Bigg\} \div \\
&\div \mathrm{tg}\,\alpha[(r_j+s)^4 - 2(\nu^2+1)(r_j+s)^2 + d], \\
B'_{js} = C_j &\Bigg\{\nu\Bigg[-(2+\mu)(r_j+s)^2 - \frac{(1+\mu)(1-2\mu)}{(1-\mu)}(r_j+s) + \\
&+ \nu^2 - \left(\frac{2\mu}{1-\mu}\right)\Bigg] d'_{js} + \nu\Bigg[-2(2+\mu)(r_j+s) - \\
&- \frac{(1+\mu)(1-2\mu)}{(1-\mu)}\Bigg] d_{js} - 4(r_j+s)[(r_j+s)^2 - (\nu^2+1)] \times \\
&\times \frac{\nu\left[-(2+\mu)(r_j+s)^2 - \frac{(1+\mu)(1-2\mu)}{(1-\mu)}(r_j+s) + \nu^2 - \left(\frac{2\mu}{1-\mu}\right)\right]}{[(r_j+s)^4 - 2(\nu^2+1)(r_j+s)^2 + d]} d_{js}\Bigg\} \div \\
&\div \mathrm{tg}\,\alpha[(r_j+s)^4 - 2(\nu^2+1)(r_j+s)^2 + d]
\end{aligned}\right\} \tag{21}
$$

(j= 1,2, and 3 -- when ν is an integer)
(s= 0,2,4,6,8,...)

We see from the above equation, the series of $X_j(x)$ and $Y_j(x)$ are also convergent. /141

The general solution for the vibration of a conical shell is found by the above method. The ratio between the eight constants C_j is determined by the eight boundary conditions, and the eigenfrequencies can be found. The exact solution can be obtained in this way. However, from the engineering view point, the calculations are obviously too complicated, unless we use high speed computors. Therefore, we shall later introduce a simplified calculation method. But we must first discuss the treatment of the boundary conditions at the vertex of a whole conical shell.

4. Treatment of the Boundary Conditions at the Vertex

We already know that the thin shell theory equations are not valid in the vicinity of the vertex of a whole conical shell. It has been stated previously that the effect of the vertex on the characteristics of the conical shell as a whole is very small. The boundary conditions involve the local displacements and forces on the vertex, which warrant further discussion.

Usually there are two different cases when prescribing the boundary conditions at the vertex: (a) Axial displacement or any displacements not allowed (restricted vertex); (b) Axial displacement allowed (free vertex). For clarity, we shall not discuss the case when arbitrary displacements are allowed, for this case is basically similar to the free vertex case mentioned above.

For case (a), note that only the axial displacement is restricted. Due to the requirement that the displacement at the vertex should be single valued, we also set

$$u = v = w = 0, \quad \frac{\partial w}{\partial x} = 0, \quad \text{at: } x = 0 \tag{22}$$

which is equivalent to inelastic fixed end conditions. From the solutions of $Z_j(x)$ and $X_j(x)$, the properties of $Y_j(x)$ can be obtained, where we must make $C_2 = C_4 = C_6 = C_8 = 0$. In this case, in general, it is not necessary to consider the correctness of the above solutions at the vertex because qualitative results are sufficient. (As for the axisymmetric case when $n = 0$ or $\nu = 0$, we must start from the form of the equations and the solutions. This will be discussed elsewhere). Then the characteristic equations become a fourth order determinant which is determined by the boundary conditions at the base.

For case (b), using the condition that the displacement of the vertex be single valued, the boundary conditions are written as

$$\text{At } x = 0\text{:} \quad w = u \operatorname{tg} \alpha, \quad v = \frac{\partial w}{\partial x} = 0, \quad N_x \cos\alpha + Q_x \sin\alpha = 0 \text{ or } P, \tag{23}$$

where N_x is the tension in the x direction, Q_x is the lateral force, and P is the external force in the axial direction. Usually there is no outside force at the vertex, so we shall take $P = 0$ from now on. Because of the inaccuracy of the equations and the solutions near the vertex (in the vicinity of x = 0), when making quantitative calculations, we should reconsider the treatment of the boundary conditions. According to the previous result that the effect on the characteristic values of the body as a whole is minute, owing to the comparatively small volume and energy of the region near the vertex, we suggest that instead of applying the boundary conditions (23) at x = 0, we apply them at $x = x_s$:

$$w = u \operatorname{tg} \alpha, \quad v = \frac{\partial w}{\partial x} = 0,$$

$$\text{At } x = x_\varepsilon: \quad \int_0^{2\pi} (N_x \cos\alpha + Q_x \sin\alpha) x_\varepsilon \sin\alpha \, d\theta = -m_\varepsilon \left[\frac{\partial^2}{\partial t^2} (u \cos\alpha + w \sin\alpha) \right]. \tag{24}$$

x_s is a small value (compared to the total length x_1). Its value is determined by the generally accepted limit $h/x \operatorname{tg} \alpha \leqq 1/25$ for the validity of the equations of thin shell theory. We take

$$x_\varepsilon = 25\, h/\operatorname{tg}\alpha;$$

where in equation (24) m_ε is the mass of the vertex region, $m_\varepsilon = (\rho h/g)\pi x_\varepsilon^2 \sin\alpha$. This kind of treatment is reasonable. From the character of the vibrational mode function (the product of x^{r_j} with the curve in Figure 1 and the vibrational mode shown in Figure 4), we see than there is almost no strain in the vicinity of the vertex. That is, the displacements of individual points in the vertex region are almost exactly equal to the displacement of vertex itself (all zero or some constant value). Therefore, we can regard the vertex region (the region whose length is x_ε) as inelastic. This is very similar to the substitution of the displacement at $x = x_\varepsilon$ for the boundary condition at x = 0. At the same time, we can simplify the last condition in equation (24), i.e., since the vertex region is considered inelastic, we can reduce the integral; in addition, we neglect the small mass m_ε and rewrite the condition as

$$\text{at } x = x_\varepsilon: \quad N_x \cos\alpha + Q_x \sin\alpha = 0$$

Expressed in terms of displacements, and using the assumption of inelasticity on the other three conditions, the above equations can be written as

$$\text{at } x = x_\varepsilon: \quad \frac{\partial u}{\partial x} - \frac{h^2 \operatorname{tg}\alpha}{12} \left(\frac{\partial^3 w}{\partial x^3} + \frac{1}{x_\varepsilon} \frac{\partial^2 w}{\partial x^2} \right) = 0, \tag{25}$$

or approximately written as $\frac{\partial u}{\partial x} = 0$, because $\frac{h^2 \operatorname{tg}\alpha}{12}$ is a small value, especially when the angle α is small. Finally, the results from the application of thin shell theory indicate the effect of an approximate boundary condition on the characteristics of vibration is rather small. Therefore, the above suggested treatment can be accepted. The solution to the problem is similar to that of the truncated conical shell. Of course, as discussed in Section II, the above considerations are not applicable to shells with very small cone angle ($\alpha < 5° - 10°$) or very short stubby shells, In such a case we should in general consider those as variable cross section beams and calculate the bending vibrations.

5. Simplified Calculations and the Parallel Springs Concept

We have neglected the effects of the longitudinal and the circumferential displacements u, v on the change of curvature and the torsional rate. Thus the third equation in equations (1) contains the terms for the internal moments due to bending and torsion, while the first and the second equations contain only the terms for the internal forces due to membrane tension. This characteristic implies that if we separate the vibration problem into two parts, i.e., using no-moment theory (membrane theory) and pure moment theory (equivalent to thin plate bending theory), and combine the two, the calculations would be greatly simplified. Pure bending theory means that we neglect the membrane resistance and consider only the resistance to bending moments. The equation is composed of the moment terms and the inertia terms in the third equation. This form actually utilizes the concepts of thin plate bending theory (neglected middle plane distortion — the displacements u, v) to compute the results for the shell. Thus the calculations are very simple.

Now we explain the underlying physical concepts for the above simplified method. Let us make a physical model for an arbitrary element on the shell: separate the elastic restoring forces due to distortion into two separate portions, one is the membrane resistance, the other is the resistance to bending moments. They are schematically represented by the springs 1 and 2 in the following figure (the spring constants k_1 and k_2 represent the elasticity of the above two portions). The element of the shell material is represented by the mass m in Figure 2. Then we regard the whole shell as a system with infinite degrees of freedom, i.e., composed of an infinite number of these single-degree-of-freedom systems. The spring constants are functions of the coordinates.

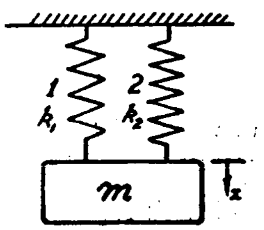

Figure 2.

The differential equation (without damping) for the parallel spring system shown in Figure 2 is /143

$$k_1 x + k_2 x = -m\frac{d^2x}{dt^2}, \tag{26}$$

Its eigenfrequency is

$$\omega = \sqrt{\frac{(k_1+k_2)}{m}} = \sqrt{\frac{k_1}{m}+\frac{k_2}{m}} = \sqrt{\omega_1^2+\omega_2^2}, \tag{27}$$

where ω_1 is the eigenfrequency of mass m with spring 1 alone, and ω_2 is the eigenfrequency for mass m with spring 2 alone.

For a general elastic body, if its equations of motion can be written in the form of equation (26), we have

$$K_1(\xi) + K_2(\xi) = M\left(\frac{\partial^2\xi}{\partial t^2}\right), \tag{28}$$

where $K_1(\xi)$ and $K_2(\xi)$ are linear functions of the variable ξ or its derivatives.

Then, under certain conditions, its eigenfrequency can be calculated in a manner similar to equation (27). For instance, this method can calculate very accurately(1) the natural frequency of a simply supported beam under tension T (addition of string vibration and lateral vibration of the beam).

Note that when the results for the vibrational mode function, which satisfy all the boundary conditions, are exactly the same when derived from two entirely different theories, then the solutions must be correct(2). One example is the above mentioned horizontal beam problem. If the vibration modes are similar, then very good approximate solutions can be obtained, such as in the case of the circular cylindrical shell. In the case of the conical shell this is also true. Because the solutions for both theories are in the form of power series (similar characteristics), the vibration modes are similar. From this discussion, we note that the physical model of the parallel springs system does describe the actual situation, and the above simplification method can be used. We should also point out that this method can be used to calculate other problems connected with the stability or vibration of shells in general.

Obviously, the above theory can also be explained in terms of energy. However, this will not be discussed in this paper.

According to this method, the problem can be decomposed into the following two sets of equations:

$$\left.\begin{aligned}&\frac{\partial^2 u}{\partial x^2}+\frac{1}{x}\frac{\partial u}{\partial x}-\frac{u}{x^2}+\left(\frac{1+\mu}{2}\right)\frac{1}{x\sin\alpha}\frac{\partial^2 v}{\partial x\,\partial\theta}-\left(\frac{3-\mu}{2}\right)\frac{1}{x^2\sin\alpha}\frac{\partial v}{\partial\theta}+\\&\quad+\left(\frac{1-\mu}{2}\right)\frac{1}{x^2\sin^2\alpha}\frac{\partial^2 u}{\partial\theta^2}+\frac{w}{x^2\,\mathrm{tg}\,\alpha}-\frac{\mu}{x\,\mathrm{tg}\,\alpha}\frac{\partial w}{\partial x}=\frac{\rho(1-\mu^2)}{gE}\frac{\partial^2 u}{\partial t^2},\\&\left(\frac{1-\mu}{2}\right)\frac{\partial^2 v}{\partial x^2}+\left(\frac{1-\mu}{2}\right)\frac{1}{x}\frac{\partial v}{\partial x}-\left(\frac{1-\mu}{2}\right)\frac{v}{x^2}+\left(\frac{1+\mu}{2}\right)\frac{1}{x\sin\alpha}\frac{\partial^2 u}{\partial x\,\partial\theta}+\\&\quad+\left(\frac{3-\mu}{2}\right)\frac{1}{x^2\sin\alpha}\frac{\partial u}{\partial\theta}+\frac{1}{x^2\sin^2\alpha}\frac{\partial^2 v}{\partial\theta^2}-\frac{\cos\alpha}{x^2\sin^2\alpha}\frac{\partial w}{\partial\theta}=\frac{\rho(1-\mu^2)}{gE}\frac{\partial^2 v}{\partial t^2},\\&\frac{\mu}{x\,\mathrm{tg}\,\alpha}\frac{\partial u}{\partial x}+\frac{u}{x^2\,\mathrm{tg}\,\alpha}+\frac{\cos\alpha}{x^2\sin^2\alpha}\frac{\partial v}{\partial\theta}-\frac{w}{x^2\,\mathrm{tg}^2\,\alpha}=\frac{\rho(1-\mu^2)}{gE}\frac{\partial^2 w}{\partial t^2};\end{aligned}\right\}\quad(29)$$

$$\nabla^2\nabla^2(w)+\frac{12\rho(1-\mu^2)}{gEh^2}\frac{\partial^2 w}{\partial t^2}=0.\quad(30)$$

(1) The equation of motion and the eigenfrequency of a simply supported beam with tension are as follows:

$$-T\frac{\partial^2 y}{\partial x^2}+EI\frac{\partial^4 y}{\partial x^4}=-m\frac{\partial^2 y}{\partial t^2}\qquad\text{and}\qquad\omega=\frac{n^2\pi^2}{l^2}\sqrt{\frac{EI}{m}\left(1+\frac{Tl^2}{n^2\pi^2 EI}\right)}\quad(n=1,2,3,\cdots).$$

The equation of motion corresponds to equation (28). Here EI is the rigidity constant, and m is the mass per unit length. Calculating separately from $T\frac{\partial^2 y}{\partial x^2}=m\frac{\partial^2 y}{\partial t^2}$ and $EI\frac{\partial^4 y}{\partial x^4}=-m\frac{\partial^2 y}{\partial t^2}$ we can obtain $\omega_1=\frac{n\pi}{l}\sqrt{\frac{T}{m}}$, $\omega_2=\frac{n^2\pi^2}{l^2}\sqrt{\frac{EI}{m}}$, from which equation (27) can be shown to be satisfied.

(2) The details of the proof shall be discussed in another paper.

The solution of these equations shall be discussed in the following two sections. /144

When using this method, we can divide the total boundary conditions (in general eight) into two sets, corresponding to the two sets of separated equations. In the membrane theory calculations, the tangential component of the displacements u, v in the middle plane must be satisfied on the boundaries (four conditions). In the pure bending theory calculations, the lateral component of the displacement w must be satisfied on the boundaries (four conditions). Of course, this kind of treatment relaxes the restrictions, because in the membrane theory calculations the lateral component of the displacement w does not necessarily satisfy the total boundary conditions. But this discrepancy is not large. Besides, in the bending theory, the arbitrariness of the boundary conditions on u and v is also questionable. However, according to the character of these separate calculations, these problems are all secondary, and should not induce large errors. Furthermore, experience in thin shell theory research tells us the effect due to slight boundary condition deviations is not large. Thus the above treatment is acceptable.

To check the accuracy of the above method, we calculated the vibrational characteristics of the case when $\alpha = 0$, or the circular cylindrical shell, because there have been more studies for comparison in this case.

6. Calculation for the Vibration of a Circular Cylindrical Shell

Using the above method, the calculations of the circular cylindrical shell can be obtained from equations (29) and (30) by setting $\alpha = 0$:

$$\left.\begin{aligned}&\frac{\partial^2 u}{\partial x^2}+\left(\frac{1-\mu}{2}\right)\frac{1}{a^2}\frac{\partial^2 u}{\partial\theta^2}+\left(\frac{1+\mu}{2}\right)\frac{1}{a}\frac{\partial^2 v}{\partial x\partial\theta}-\frac{\mu}{a}\frac{\partial w}{\partial x}=\frac{\rho(1-\mu^2)}{gE}\frac{\partial^2 u}{\partial t^2},\\&\left(\frac{1-\mu}{2}\right)\frac{\partial^2 v}{\partial x^2}+\frac{1}{a^2}\frac{\partial^2 v}{\partial\theta^2}+\left(\frac{1+\mu}{2}\right)\frac{1}{a}\frac{\partial^2 u}{\partial x\partial\theta}-\frac{1}{a^2}\frac{\partial w}{\partial\theta}=\frac{\rho(1-\mu^2)}{gE}\frac{\partial^2 v}{\partial t^2},\\&\frac{\mu}{a}\frac{\partial u}{\partial x}+\frac{1}{a^2}\frac{\partial v}{\partial\theta}-\frac{w}{a^2}=\frac{\rho(1-\mu^2)}{gE}\frac{\partial^2 w}{\partial t^2};\end{aligned}\right\}\quad(31)$$

$$\nabla_0^2\nabla_0^2(w)+\frac{12\rho(1-\mu^2)}{gEh^2}\frac{\partial^2 w}{\partial t^2}=0.\quad(32)$$

I. Membrane Theory Calculations

For simplicity, we still neglect the terms due to the tangential components of the inertia forces. The solutions of u, v, w are assumed to be of the form:

$$\left.\begin{aligned}u&=U_1(x)\cos n\theta\sin\omega_1 t,\\v&=V_1(x)\sin n\theta\sin\omega_1 t,\\w&=W_1(x)\cos n\theta\sin\omega_1 t.\end{aligned}\right\}\quad(33)$$

Substitution into equation (31) and separation of the variables results in

$$\nabla^2_{0x}\nabla^2_{0x}U_1 - \frac{\mu}{a}\frac{d^3W_1}{dx^3} - \frac{n^2}{a^3}\frac{dW_1}{dx} = 0, \tag{34a}$$

$$\nabla^2_{0x}\nabla^2_{0x}V_1 + (2+\mu)\frac{n}{a^2}\frac{d^2W_1}{dx^2} - \frac{n^3}{a^4}W_1 = 0, \tag{34b}$$

$$\frac{d^4W_1}{dx^4} + b_1\frac{d^2W_1}{dx^2} - b_2W_1 = 0, \tag{34c}$$

where $b_1 = 2n^2\frac{\rho\omega_1^2}{gE}\Big/\left(1 - \frac{\rho\omega_1^2}{gE}a^2\right)$, $b_2 = \frac{n^4}{a^2}\frac{\rho\omega_1^2}{gE}\Big/\left(1 - \frac{\rho\omega_1^2}{gE}a^2\right)$. Note that for the few lowest order eigenfrequencies which are usually calculated, $\frac{\rho\omega_1^2}{gE}a^2 \ll 1$, holds. We can take $b_1 \approx 2n^2\frac{\rho\omega_1^2}{gE}$, $b_2 \approx \frac{n^4}{a^2}\frac{\rho\omega_1^2}{gE}$. Because in most cases $\frac{gE}{a^2\rho\omega^2_{\min}} > 100$, we can even consider $\frac{1}{a}\sqrt{\frac{gE}{\rho\omega_1^2}} \gg 1$, to be true. Thus the solution for $W_1(x)$ is

$$W_1(x) = C_1\cos\lambda_1x + C_2\sin\lambda_1x + C_3\,\mathrm{ch}\,\lambda_1x + C_4\,\mathrm{sh}\,\lambda_1x, \tag{35}$$

where C_1, C_2, C_3, C_4 are undetermined constants, and $\lambda_1 = n\sqrt[4]{\frac{\rho\omega_1^2}{a^2gE}}$. From equations (34a) and (34b) we find

$$\left.\begin{aligned} U_1(x) &= A_1\sin\lambda_1x + A_2\cos\lambda_1x + A_3\,\mathrm{sh}\,\lambda_1x + A_4\,\mathrm{ch}\,\lambda_1x,\\ V_1(x) &= B_1\cos\lambda_1x + B_2\sin\lambda_1x + B_3\,\mathrm{ch}\,\lambda_1x + B_4\,\mathrm{sh}\,\lambda_1x,\end{aligned}\right\} \tag{36}$$

where

$$\left.\begin{aligned} &\begin{matrix}A_1\\A_2\end{matrix}\Big\} = \pm\frac{\delta_{12}}{R_+}\Big\{\begin{matrix}C_1\\C_2\end{matrix}, \qquad \begin{matrix}A_3\\A_4\end{matrix}\Big\} = \frac{\delta_{34}}{R_-}\Big\{\begin{matrix}C_3\\C_4\end{matrix},\\ &\begin{matrix}B_1\\B_2\end{matrix}\Big\} = \frac{\eta_{12}}{R_+}\Big\{\begin{matrix}C_1\\C_2\end{matrix}, \qquad \begin{matrix}B_3\\B_4\end{matrix}\Big\} = -\frac{\eta_{34}}{R_-}\Big\{\begin{matrix}C_3\\C_4\end{matrix},\end{aligned}\right\} \tag{37}$$

and

$$\begin{matrix}\delta_{12}\\\delta_{34}\end{matrix}\Big\} = \frac{\lambda_1}{a}\left(\mu\lambda_1^2 \mp \frac{n^2}{a^2}\right), \qquad \begin{matrix}\eta_{12}\\\eta_{34}\end{matrix}\Big\} = \frac{n}{a^2}\left[(2+\mu)\lambda_1^2 \pm \frac{n^2}{a^2}\right],$$

$$\begin{matrix}R_+\\R_-\end{matrix}\Big\} = \left(\lambda_1^4 \pm 2\frac{n^2}{a^2}\lambda_1^2 + \frac{n^4}{a^4}\right).$$

Using the corresponding boundary conditions for the membrane theory equations, we can derive the characteristic equation and solve for the eigenfrequencies. For instance, in the case of freely supported ends: at x = 0 and x = ℓ, $v = \frac{\partial u}{\partial x} = 0$. We find

$$\omega_1 = \left(\frac{m\pi}{nl}\right)^2\sqrt{\frac{a^2E}{\rho/g}} \quad (m = 1, 2, 3, \cdots). \tag{38}$$

We see that this is similar to the vibration of the simply supported beam. For the case when both ends are rigidly supported, at x = 0 and x = ℓ, u = v = 0. The characteristic equation is

$$1 + \frac{1}{2}\left(\beta_1 - \frac{1}{\beta_1}\right)\sin\lambda_1 l \cdot \operatorname{sh}\lambda_1 l = \cos\lambda_1 l \cdot \operatorname{ch}\lambda_1 l, \tag{39}$$

where $\beta_1 = \frac{\delta_{34}\eta_{12}}{\delta_{12}\eta_{34}}$. From this we can calculate the eigenfrequencies.

The above results can only be applied to a few low order eigenfrequencies. If we want to calculate the higher order eigenfrequencies, we must abandon the assumption $\frac{1}{a}\sqrt{\frac{gE}{\rho\omega_1^2}} \gg$ and retain $\frac{gE}{a^2\rho\omega_1^2} \gg 1$. Take

$$W_1(x) = C_1\cos\lambda_2' x + C_2\sin\lambda_2' x + C_3\operatorname{ch}\lambda_1' x + C_4\operatorname{sh}\lambda_1' x,$$

where

$$\lambda_1' = n\sqrt{\frac{\rho\omega_1^2}{gE}\left[\frac{1}{a}\sqrt{\frac{gE}{\rho\omega_1^2}} - 1\right]},\quad \lambda_2' = n\sqrt{\frac{\rho\omega_1^2}{gE}\left[\frac{1}{a}\sqrt{\frac{gE}{\rho\omega_1^2}} + 1\right]}.$$

We see that the above calculations are similar to the calculations of the vibration of an elastic beam, which is relatively simple. /146

II. Pure Bending Theory Calculations

For the solution of equation (32), let w be of the form

$$w = W_2(x)\cos n\theta \sin\omega_2 t. \tag{40}$$

Substituting into equation (32) and separating the variables, we obtain

$$\nabla_{0x}^2\nabla_{0x}^2(W_2) - k_2^4 W_2 = 0 \quad \left(k_2^4 = \frac{12\rho(1-\mu^2)}{gEh^2}\omega_2^2\right), \tag{41}$$

The solution is

$$W_2(x) = A\cos\sqrt{k_2^2 - \frac{n^2}{a^2}}\,x + B\sin\sqrt{k_2^2 - \frac{n^2}{a^2}}\,x + C\operatorname{ch}\sqrt{k_2^2 + \frac{n^2}{a^2}}\,x + D\operatorname{sh}\sqrt{k_2^2 + \frac{n^2}{a^2}}\,x. \tag{42}$$

For the case when both ends are freely supported, where at x = 0 and x = ℓ, $w = 0,\ \frac{\partial^2 w}{\partial x^2} + \mu\left(\frac{\partial^2 w}{a^2\partial\theta^2} + \frac{1}{a^2}\frac{\partial v}{\partial\theta}\right) = 0 \to \frac{\partial^2 w}{\partial x^2} = 0;$ we find

$$\omega_2 = \left(\frac{m^2\pi^2}{l^2} + \frac{n^2}{a^2}\right)\sqrt{\frac{Eh^2 g}{12(1-\mu^2)\rho}} \quad (m = 1, 2, 3, \cdots). \tag{43}$$

For the case when both ends are rigidly supported, where at x = 0 and x = ℓ, $w = \frac{\partial w}{\partial x} = 0$, the characteristic equation is

$$1+\frac{1}{2}\left(\beta_2-\frac{1}{\beta_2}\right)\sin\sqrt{k_2^2-\frac{n^2}{a^2}}\,l\cdot\operatorname{sh}\sqrt{k_2^2+\frac{n^2}{a^2}}\,l=$$
$$=\cos\sqrt{k_2^2-\frac{n^2}{a^2}}\,l\cdot\operatorname{ch}\sqrt{k_2^2+\frac{n^2}{a^2}}\,l, \tag{44}$$

where $\beta_2=\sqrt{\left(k_2^2+\frac{n^2}{a^2}\right)\Big/\left(k_2^2-\frac{n^2}{a^2}\right)}$. From the solution of ω_2 we can find the eigenfrequency from the equation $\omega=\sqrt{\omega_1^2+\omega_2^2}$.

We can see from the above analysis that the vibrational mode functions derived from these two separate theories have the same character and form. This agrees with the requirement of the calculation method. Besides, the characteristic equations under the corresponding boundary conditions are also similar. Further note that when using this method the work required for the calculations is equivalent to that for the vibrating elastic beam. Its simplicity is evident.

To check the accuracy of the eigenfrequency $(\omega=\sqrt{\omega_1^2+\omega_2^2})$ calculated by this method, we will apply the results to a specific example, and compare with the existing research (Ref. 18 - 20).

Take a simply supported circular cylindrical shell with the following data:

$$a = 20 \text{ cm},\ \ell = 54 \text{ cm},\ h = 0.80 \text{ mm}$$
$$E = 2.0 \times 10^6 \text{ Kg/cm}^2,\ \rho/g = 7.95 \times 10^{-6} \text{ Kg}\cdot\text{sec}^2/\text{cm}^4,\ \mu = 0.28.$$

For different values of n, the lowest (m = 1) frequencies ($f = \omega/2\pi$) are tabulated as follows: /147

Method of Calculation and Experimental Results	Eigenfrequency $f = \omega/2\pi$ (m = 1)					
	n = 3	n = 4	n = 5	n = 6	n = 8	n = 10
Theory of Arnold and Warburton (Ref. 18)	530	329	258	-	-	-
Theory of Baron and Belich (Ref. 20)	504	320	241	235	-	-
Theory of Brelavskiy	-	-	237	225	307	476
Experiments of Brelavskiy	-	-	250	233	334	497
Present Theory	603	348	250	235	325	490

We see that the results from the above calculations are satisfactory, except the case n = 3, where the difference is larger. This is probably due to the fact that the error is larger in the thin shell theory when n is small. Note that the calculations of Baron and Bleich (Ref. 20) are based on a similar physical concept as the present paper. The difference is that the former uses an energy method and solves for the solutions using the results from the membrane theory as the approximate vibration mode, while the present paper uses analytical methods of two separate theories when seeking the solution. Since the results for the vibration mode are the same or similar, our method should be more accurate (when n > 3).

7. Calculations for the Vibrations of a Conical Shell

I. Using the No-moment Theory (Membrane Theory)

For the solutions of equations (29), we also assume the answer is in the form of equation (33). After separating the variables, and using the previously discussed assumptions and method of calculation, the following independent equation for $W_1(x)$ is obtained:

$$\Delta^4(W_1) - \Delta^2(W_1) + eW_1 - \frac{\Omega_1^2 \operatorname{tg}^2\alpha}{(1-\mu^2)} L(x^2 W_1) = 0, \tag{45}$$

where $\Omega_1^2 = \frac{\rho(1-\mu^2)}{gE}\omega_1^2.$ The relationships between U_1, V_1 and W_1 are expressed by equations (4a) and (4b).

Let the solution $W_1(x)$ be in the form of a power series:

$$W_1(x) = \sum_{j=1}^{4}\sum_{s=0}^{\infty} c_{js} x^{r_j+s}, \tag{46}$$

Substitute into equation (45), and use the condition that the coefficients of each power of the function be zero. The indices r_j and c_{js} are then found. From the definition of a series we have $c_{j,-1} = c_{j,-2} = 0$. Therefore, when $s = 0$, we find

$$r_j = r_{1,2,3,4} \approx \pm\frac{\lambda}{2}\left[\left(1+\frac{1}{2\lambda^2}\right) \pm i\left(1-\frac{1}{2\lambda^2}\right)\right]. \tag{47}$$

when ν is large (for example when $\alpha < 30°$), and because $\lambda^2 \gg 1$, we can take

$$r_1 = \frac{\lambda}{2}(1+i), \quad r_2 = \frac{\lambda}{2}(1-i), \quad r_3 = -\frac{\lambda}{2}(1+i), \quad r_4 = -\frac{\lambda}{2}(1-i). \tag{48}$$

We continue to solve for the value of c_{js}. We find $c_{j1} = c_{j3} = c_{j5} = \ldots = c_{j,2s+1} = 0$, but

$$c_{js} = \frac{\Omega_1^2 \operatorname{tg}^2\alpha}{(1-\mu^2)}\left[\frac{(r_j+s)^4 - 2(\nu^2+1)(r_j+s)^2 + d}{(r_j+s)^4 - (r_j+s)^2 + e}\right] c_{j,s-2} \quad (s = 2, 4, 6, 8, \cdots). \tag{49}$$

If we set the arbitrary constant $c_{j0} = 1$, we can define the following function

$$F_{r_j}(\nu, x\Omega_1 \operatorname{tg}\alpha) = 1 + \sum_{s=2,4,6}^{\infty} \prod\left[\frac{(r_j+s)^4 - 2(\nu^2+1)(r_j+s)^2 + d}{(r_j+s)^4 - (r_j+s)^2 + e}\right] \times \tag{50}$$

$$\times \frac{(x\Omega_1 \operatorname{tg}\alpha)^s}{(1-\mu^2)^{s/2}} = \operatorname{Re}(F_{r_j}) + i\cdot \operatorname{Im}(F_{r_j}). \qquad (50)\ \text{(cont.)}$$ /148

Then the lateral vibrational mode function $W_1(x)$ can be written as c_j

$$W_1(x) = \sum_{j=1}^{4} c_j x^{r_j} F_{r_j}(\nu, x\Omega_1 \operatorname{tg}\alpha), \qquad (51)$$

where c_j is an undetermined constant (in general complex). The convergence of F_{rj} can be determined from the Cauchy criteria

$$\lim_{s\to\infty}\left[\frac{(r_j+s)^4-2(\nu^2+1)(r_j+s)^2+d}{(r_j+s)^4-(r_j+s)^2+e}\right]\frac{(x\Omega_1 \operatorname{tg}\alpha)^2}{(1-\mu^2)} = \frac{(x\Omega_1 \operatorname{tg}\alpha)^2}{(1-\mu^2)}, \qquad (52)$$

When $\frac{(x\Omega_1 \operatorname{tg}\alpha)^2}{(1-\mu^2)} < 1$, the function F_{rj} is convergent. From the calculations, this requirement is satisfied for most low-order eigenfrequencies. Because usually it is not necessary to calculate the higher order eigenfrequencies, the above solutions are sufficient for practical purposes. If we only want the first two lowest order frequencies, the above series converges so fast that two or three terms already result in three-digit accuracy. This is the main advantage of such a method. It avoids the difficulties encountered in solving for the exact solutions and therefore possesses a certain practical value.

After $W_1(x)$ is found, it is not difficult to obtain $U_1(x)$ and $V_1(x)$:

$$\left.\begin{aligned} U_1(x) &= \sum_{j=1}^{4} c_j x^{r_j} G_{r_j}(\nu, x\Omega_1 \operatorname{tg}\alpha), \\ V_1(x) &= \sum_{j=1}^{4} c_j x^{r_j} H_{r_j}(\nu, x\Omega_1 \operatorname{tg}\alpha), \end{aligned}\right\} \qquad (53)$$

where

$$\left.\begin{aligned} G_{r_j}(\nu, x\Omega_1 \operatorname{tg}\alpha) &= \frac{1}{\operatorname{tg}\alpha}\left\{\left[\frac{\mu r_j^3 - r_j^2 + (\nu^2-\mu)r_j - \left(\frac{2\mu}{1-\mu}\right)\nu^2 + 1}{r_j^4 - 2(\nu^2+1)r_j^2 + d}\right] + \right. \\ &+ \sum_{s=2,4,6}^{\infty}\left[\frac{\mu(r_j+s)^3-(r_j+s)^2+(\nu^2-\mu)(r_j+s)-\left(\frac{2\mu}{1-\mu}\right)\nu^2+1}{(r_j+s)^4-2(\nu^2+1)(r_j+s)^2+d}\right]\times \\ &\times \left.\prod\left[\frac{(r_j+s)^4-2(\nu^2+1)(r_j+s)^2+d}{(r_j+s)^4-(r_j+s)^2+e}\right]\frac{(x\Omega_1 \operatorname{tg}\alpha)^s}{(1-\mu^2)^{s/2}}\right\} = \\ &= \operatorname{Re}(G_{r_j}) + i\cdot \operatorname{Im}(G_{r_j}), \\ H_{r_j}(\nu, x\Omega_1 \operatorname{tg}\alpha) &= \frac{\nu}{\operatorname{tg}\alpha}\left\{\left[\frac{-(2+\mu)r_j^2 - \frac{(1+\mu)(1-2\mu)}{(1-\mu)}r_j + \nu^2 - \frac{2\mu}{1-\mu}}{r_j^4 - 2(\nu^2+1)r_j^2 + d}\right] + \right. \\ &+ \sum_{s=2,4,6}^{\infty}\left[\frac{-(2+\mu)(r_j+s)^2 - \frac{(1+\mu)(1-2\mu)}{(1-\mu)}(r_j+s) + \nu^2 - \left(\frac{2\mu}{1-\mu}\right)}{(r_j+s)^4-2(\nu^2+1)(r_j+s)^2+d}\right]\times \\ &\times \left.\prod\left[\frac{(r_j+s)^4-2(\nu^2+1)(r_j+s)^2+d}{(r_j+s)^4-(r_j+s)^2+e}\right]\frac{(x\Omega_1 \operatorname{tg}\alpha)^s}{(1-\mu^2)^{s/2}}\right\} = \\ &= \operatorname{Re}(H_{r_j}) + i\cdot \operatorname{Im}(H_{r_j}). \end{aligned}\right\} \qquad (54)$$

Obviously both series are convergent. Applying the boundary conditions to the solution, a characteristic equation can be obtained (in general, a fourth-order determinant) from which we can solve for the eigenfrequencies. There are four boundary conditions when using the no-moment theory for a truncated conical shell. These are represented by the displacements u, v or the tension. For example, in the case when both ends are fixed, we have /149

$$\text{at } x = x_0 \text{ and } x = x_1,\ u = v = 0.$$

For a whole conical shell, we must take $c_3 = c_4 = 0$.

We must point out that since r_1 and r_2, r_3 and r_4 are complex conjugates, the following conjugate relationship between the vibrational mode functions can be proven:

$$\left.\begin{aligned}
&\mathrm{Re}(F_{r_1})=\mathrm{Re}(F_{r_2}),\ \mathrm{Re}(F_{r_3})=\mathrm{Re}(F_{r_4}),\ \mathrm{Im}(F_{r_1})=-\mathrm{Im}(F_{r_2}),\ \mathrm{Im}(F_{r_3})=-\mathrm{Im}(F_{r_4});\\
&\mathrm{Re}(G_{r_1})=\mathrm{Re}(G_{r_2}),\ \mathrm{Re}(G_{r_3})=\mathrm{Re}(G_{r_4}),\ \mathrm{Im}(G_{r_1})=-\mathrm{Im}(G_{r_2}),\ \mathrm{Im}(G_{r_3})=-\mathrm{Im}(G_{r_4});\\
&\mathrm{Re}(H_{r_1})=\mathrm{Re}(H_{r_2}),\ \mathrm{Re}(H_{r_3})=\mathrm{Re}(H_{r_4}),\ \mathrm{Im}(H_{r_1})=-\mathrm{Im}(H_{r_2}),\ \mathrm{Im}(H_{r_3})=-\mathrm{Im}(H_{r_4}).
\end{aligned}\right\} \tag{55}$$

These conjugate relationships greatly simplify the calculations of the functions and also simplify the solutions of the characteristic equations.

II. Using Pure Moment Theory (Bending Theory)

It is easy to see that solving equation (30) will be much simpler than when using membrane theory. The solution is still a Bessel function. The solution for the lateral displacement w is also assumed to be in the form of equation (40). Substituting into equation (30) we have

$$\nabla_x^2\nabla_x^2(W_2) = k^4 W_2, \text{ or}$$

$$\left[\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} + \left(k^2 - \frac{\nu^2}{x^2}\right)\right]\left[\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - \left(k^2 + \frac{\nu^2}{x^2}\right)\right]W_2 = 0 \tag{56}$$

The solution is

$$W_2(x) = AJ_\nu(kx) + BY_\nu(kx) + CI_\nu(kx) + DK_\nu(kx), \tag{57}$$

where $J_\nu(kx)$, $Y_\nu(kx)$, $I_\nu(kx)$, $K_\nu(kx)$ are Besses functions of order ν. The boundary conditions are similar to those for the bending of a thin plate, e.g.

for fixed edges at $x = x_0$ and $x = x_1$, $w = \frac{\partial w}{\partial x} = 0$,

for simply supported edges: at $x = x_0$ and $x = x_1$, $w = M_x = 0$,

for free edges: at $x = x_0$ and $x = x_1$, $Q_x = M_x = 0$.

For the restriction of the vertex in the case of a whole shell, we should take $B = D = 0$. ω_2 can be found from the characteristic equation. Finally, the eigenfrequencies are obtained from $\omega^2 = \omega_1^2 \quad \omega_2^2$.

8. A Practical Example and Discussions

In the following we shall calculate the eigenfrequencies and the vibration mode of a whole conical shell with $\alpha = 30°$. We shall then compare with the experimental results and discuss them. The data are taken to be as follows:

$$\alpha = 30°,\ x_1 = 30 \text{ cm},\ h = 0.33,\ 0.71 \text{ and } 1.64 \text{ mm},$$

$$E = 2.05 \times 10^6 \text{ Kg/cm}^2,\ \rho/g = 7.95 \times 10^{-6} \text{ Kg}\cdot\text{sec}^2/\text{cm}^4,\ \mu = 0.30.$$

Consider the case when a restricted vertex and a fixed base are the boundary conditions. Using the method developed in the two previous sections, we calculate the lowest few eigenfrequencies corresponding to $n = 3, 4, 5, \ldots, 8$. The results are shown in Figure 3 (where ω_1 and $\omega_2(\omega'_2, \omega''_2)$ are the results from no-moment theory and pure moment theory respectively, and $\omega(\omega', \omega'')$ is the final combined answer). We see that ω_1 varies inversely as a function of n, while ω_2 is just the opposite. This is to be expected. The number n which corresponds to the lowest eigenfrequency varies with the shell thickness (or the ratio between shell thickness and the diameter). When the other dimensions are the same, then the larger the h, the smaller the n, which corresponds to the lowest frequency. This is because from no-moment theory, ω_1 is independent of the thickness h, but from pure moment theory ω_2 is proportional to h.

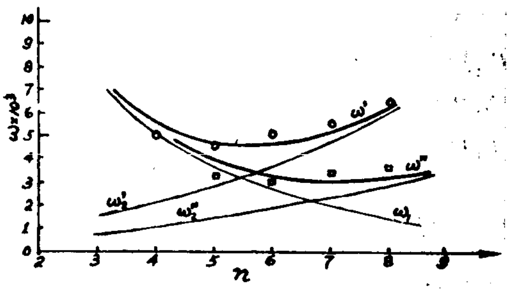

Figure 3.

ω' -- basic frequency for a $h = 0.71$ mm shell; ω'' -- basic frequency for a $h = 0.33$ mm shell; O -- experimental results for ω'; □ -- experimental results for ω''.

From Figure 3, we notice that, when vibrating under the lowest eigenfrequency, the effect of membrane tension (or the corresponding strain energy) is just equal to the effect of the moments, for both curves of ω_1 and ω_2 are concave. /150

This rule may be of some value in general shell research, for it is obviously not limited to conical shells and can be applied to general shells.

We made some experiments to check the results. The shell is made from thin plates and shaped on a mold. The seam is percussively welded (until the shell is watertight), and the specimen has a specific degree of imperfection. The bottom is welded to a thick flange. The vertex and the bottom are both attached firmly to a frame. The vibrations are induced both by the inertia activation method and by the electromagnet activation method. The results by the two methods are essentially similar, as shown in Figure 3. We see that agreement between theory and the experiments are generally good. According to the basic concepts of shell theory, shell imperfections like initial curvature do not greatly affect its characteristics, at least when the value of n is large. This is because the initial curvature has definite effects on the membrane effect (decreases its effect, but not very much), but has no effect on the pure bending effect. The experimental results in Figure 3 are a little low for small n, which verifies this prediction. Therefore, we can at least use the experiments to check the accuracy of the calculations at large n.

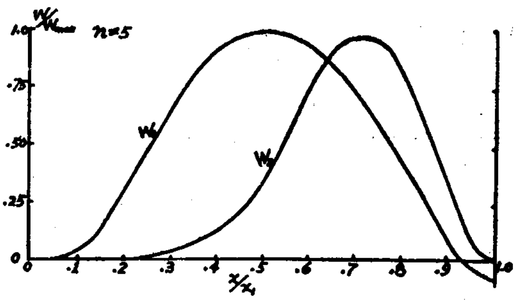

Figure 4.

Figure 4 shows the vibration mode solutions from the no-moment theory and the pure moment theory when n = 5 (corresponds to the lowest ω'). Their characteristics are similar but their form is not the same. Therefore, the results are approximate and the values are lower than the exact solution. From the physical view point, the difference of these two vibration modes means relaxed restraints. In addition, the boundary conditions are reduced, so we predict the results are on the low side. This conclusion can also be proven mathematically, which will not be discussed here. From the predictions and the preliminary experimental results, we expect the actual vibration mode lies between the two curves in Figure 4. The exact situation still requires further study. The accuracy of the calculations is obviously related to the similarities of the two vibration modes. However, from the above example and the previous calculations for other vibration problems, the effect is not large -- i.e., although the similarities are poor, it does not mean there are very large errors (in general, smaller than 10 - 20%, but sometime may reach 30%). We should note that the present method is similar to the concepts of the Dunkerley method, which is well known in the calculations of the vibration of many degree of freedom systems. Thus, the order of magnitude of the error should also be similar (i.e., the results are accurate although the vibration modes differ). Also note that the form of the vibration mode of a whole conical shell is quite different, and the error tends to be on the large side, while for truncated conical shells, the vibration modes are very similar from the two theories, and the results are more correct.

Of the existing theories, only the research of Federhofer (Ref. 2) is able to calculate the above example. He calculated the lowest value of ω to be 12000, which when h = 1.64, corresponds to n = 3. The present theory obtains 7040 and corresponds to n = 4 (or 3). The difference is not small. We already know that the present results are on the low side. But Federhofer uses a symmetric vibrational mode function $x^2\left(1-\frac{x}{x_1}\right)^2$, which is far from reality. Therefore, his results from the Rayleigh-Ritz method are obviously much larger. We should also point out that the results usually are much larger when we attempt to solve directly with an approximate form of a power function which consists /151
of few terms. For example, in the study of the stability of conical shells, the results of Ryayamet (Ref. 21) are 50 - 80% larger [see (Ref. 22)]. The author also encountered this situation in other vibrational calculations. This problem should be studied further, both theoretically and experimentally.

Finally, we point out that our methods of decomposing the equations, if advanced mathematically, can be used to improve the similarity of the vibration modes and also the accuracy of the calculations. This area needs further study.

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